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3 March 2018.

MTS.

THH and p-adic cohomology.

I) Motivation. $X/\mathbb{C}$ sm. proj.

$$H^i(X,\mathbb{Z})\otimes\mathbb{C}\cong H^i_{dR}(X/\mathbb{C}).$$

Elements in $H_i(X,\mathbb{Z})$ give non-trivial obstruction to integrating differential forms.

Q. What about $H_i(X,\mathbb{Z}/p)$?

A. Use the mod p geometry of X.

Thm H. p a prime, $X/\mathbb{Z}_p$ sm. prop. Then,

$$\dim_{\mathbb{F}_p} H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p},\mathbb{F}_p)\leq \dim_{\mathbb{F}_p} H^i_{dR}(X_{\mathbb{F}_p}/\mathbb{F}_p).$$

Moreover,

$$\ell\left(H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p},\mathbb{F}_p)_{tors}\right)\leq \ell\left(H^i_{dR}(X/\mathbb{Z}_p)_{tors}\right).$$

Remarks. i) Both inequalities can be strict.

ii) Variant exists over any local field over $\mathbb{Q}_p$.

iii) Previous work in p-adic Hodge theory was restricted and a little integral when $\dim X < p-1$. This was Faltings, Caruso. In this case, get equality.

Ex. $p=2$. $X/\mathbb{Z}_2$ Enriques surface. So, $\pi_1 X_{\overline{\mathbb{Q}}_2}\cong\mathbb{Z}/2$.

Hence, $H^1(X_{\overline{\mathbb{Q}}_2},\mathbb{F}_2)\neq 0$. Hence, $H^1(X_{\mathbb{F}_2}/\mathbb{F}_2)\neq 0$.

This was known by Lang, Illusie. Regarded as a pathology.

I) Prismatic cohomology and THH.

Idea: construct a specialization between the étale theory and the dR theory.

$$H^*(X_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_p), \quad H^*_{dR}(X/\mathbb{Z}_p).$$

Deduce Thm H by semicontinuity.

Notation. $A = \mathbb{Z}_p[[u]]$. φ Frobenius, $u \mapsto u^p$.

$E(u) = u - p$.

Eisenstein polynomial.

$A/E \cong \mathbb{Z}_p$.

Thm K. For $X/\mathbb{Z}_p$ sm. proj, there exists a perfect complex $\Delta_{X/A}$ of A-modules with the following properties:

(1) $$\Delta_{X/A} \otimes_A \underset{\mathbb{Z}_p}{\underset{\shortparallel\wr}{A/E}} \simeq R\Gamma_{dR}(X/\mathbb{Z}_p)$$

(2) $$\Delta_{X/A} \otimes_A A[\tfrac{1}{u}]^\wedge_p \underset{\uparrow}{\simeq} R\Gamma_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_p) \otimes_{A} A[\tfrac{1}{u}]^\wedge_p.$$

noncanonical, but can make it so by using $A_{\inf}$

Rem. $K \Rightarrow H$.

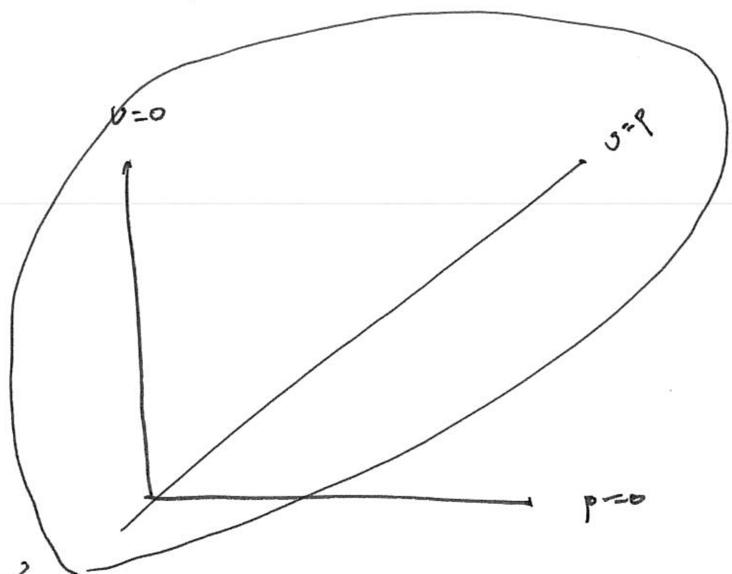
Strategy. Find a good deformation of de Rham cohomology across $A \longrightarrow A/E \cong \mathbb{Z}_p$.

Naive guess / problem. Purely alg. approach, absolute crystalline cohomology. Doesn't quite work. Only gives a deformation over $S = A[\{\frac{E^n}{n!}\}]^\wedge_p$, divided power algebra.

$\mathrm{Spec}\, A$

$\uparrow$

$\mathrm{Spec}\, S$



hits this region. Misses $p=0$, which is what governs crystalline cohomology.

Solution. Use THH.

Toy example. - $HH(\mathbb{F}_p/\mathbb{Z}_p)$ divided power algebra.

- $THH(\mathbb{F}_p)$ awesome by Bökstedt:

$$\pi_* THH(\mathbb{F}_p) \cong \mathbb{F}_p[\sigma].$$

III) Motivic filtrations on THH and its cousins, TC^-, TP.

Goal: p-complete R. Construct $\hat{\Delta}_R$ as gr^0 of a motivic filtration on $TP(R)$. Properties:

- $R/\mathbb{F}_p$ sm. $\hat{\Delta}_R \simeq R\Gamma_{crys}(R/W)$.
- mixed char. $\hat{\Delta}_R \simeq$ theory from Thm K.

<u>Ex</u>. $R/\mathbb{F}_p$ perfect. $THH_*(R) \cong R[\sigma]$. So,

$$TP_*(R) \cong W(R)[\sigma^{\pm 1}].$$

In this case, take the double-speed Postnikov filtration.

$$\hat{\Delta}_R = gr^0 TP(R) \cong \pi_0 TP(R) \cong W(R).$$

<u>Thm D</u>. THH is a sheaf for the flat topology.
So, $A \to B$ faithfully flat, then

$$THH(A) \simeq \lim_{\Delta} THH(B^{\otimes_A \bullet+1}).$$

Same for TC^-, TP, TC.

From now on, assume $p = 0$.

quasi-syntomic.

<u>Main construction</u>. $QSyn_{\mathbb{F}_p}$ - category of $\mathbb{F}_p$-algebras R

s.t. $L_{R/\mathbb{F}_p}$ has Tor-amplitude in $[0,1]$.

<u>Exs</u>. (a) perfect rings,
(b) smooth algebras over perfect,
(c) lci over perfects.

$$\mathrm{Fil}^i\, \mathrm{THH}(R) := R\Gamma(\mathrm{QSyn}_{R/}, \tau_{\geq 2i}\mathrm{THH}).$$

Similarly for cousins.

How to calculate?

1) Each quasisyntomic A admits a cover $A \to B$ with B quasiregular semiperfect.

$\uparrow$

$L_{B/\mathbb{F}_p}$ has Tor-amplitude in [redacted] $[1,1]$.
Don't have to assume φ is surjective.

Ex. $A = \mathbb{F}_p[x]$, $B = \mathbb{F}_p[x^{1/p^\infty}]$,

$$B \otimes_A B \cong \mathbb{F}_p[x^{1/p^\infty}, y^{1/p^\infty}]/(x-y).$$

2) For B qrsp, $\mathrm{THH}_*(B)$ lives in even degree, $\mathrm{THH}_{2n}(B)$ has finite filtration [redacted] w/ass. gradeds $\wedge^i L_{B/\mathbb{F}_p}[-i]$.

So, for such B, motivic filtration = double speed Postnikov.

Rmk. 1) Filtration on $\mathrm{gr}^0 TP(-)$ comes from Tate spectral sequence. This corresponds to the Nygaard filtration on $\hat{\Delta}$. This is nice because, classically have to use dRW to get this.

2) Cyclotomic Frobenius, $\phi^{tS^1} : TC^- \longrightarrow TP$. Leads to divided Frobenius on crystalline cohomology. Again, need dRW or something classically.

Cor. $R/\mathbb{F}_p$ smooth. Then, $THH(R) \xrightarrow{\varphi} THH(R)^{tC_p}$ is an $\simeq$ in large degrees.

3) $K(R)$ has the motivic filtration, Bloch, Suslin, ...

Conj (Hesselholt). $K(R) \longrightarrow TC(R)$ is compatible with a certain motivic filtration $TC(R)$ and the motivic filtration.